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The exponential map of GL(N)

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Abstract. A finite expansion of the exponential map for a $N \times N$ matrix is presented. The method uses the Cayley–Hamilton theorem for writing the higher matrix powers in terms of those for the first $N-1$. The resulting sums over the corresponding coefficients are rational functions of the eigenvalues of the matrix.

1. Introduction

In the Lie theory of groups and their corresponding algebras the exponential map is a crucial tool because it gives the connection between a Lie algebra element $H \in \mathfrak{g}$ and the corresponding Lie group element $\boldsymbol{g} \in \boldsymbol{G}$

$$\exp\colon \begin{array}{ccc} g & \rightarrow & \boldsymbol{G} \\ H & \mapsto & \boldsymbol{g} \end{array}$$

(for details see [5] and [1, 2] and references therein). In some low-dimensional cases, such as SU(2) and SO(3), the explicit expansion of the exponential map is known. Some years ago the exponential map for the Lorentz group was given by Rodrigues and Zeni [8, 9]. For the higher-dimensional groups, SU(2, 2) and O(2, 4), a method for the expansion was developed by Barut *et al* [1, 2]. The subject of the present paper is a generalization of the method developed in [1, 2] to the general linear groups GL(N). The result will be a method to calculate the exponential of a quadratic matrix H, where only rational functions of the eigenvalues of H and the first $N-1$ powers of H are involved. The key points are the Cayley–Hamilton theorem and the introduction of a multiplier m.

The organization of this paper is as follows. First, the method is shown in the low-dimensional case SU(3) which is a group occurring quite often in physics. Then the general case of the expansion of the exponential map for a $N \times N$ matrix is presented.

The only real problem that remains is the determination of the eigenvalues of the matrix H. Throughout the paper we assume that the groups GL(N) are represented as N-dimensional matrices and that the eigenvalues of H are all different if not stated otherwise (cf section 3.5).

We also consider the case of two equal eigenvalues. Then the method does also apply but the results become less simple.

Some possible applications of these results will be presented in a future paper.

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2. The exponential map for the group SU(3)

The group SU(3) is used in several branches of physics. The best known application is the model of the strong interaction (see e.g. [3]). For this reason and because it is a good exercise to follow the steps of the general method, we will show the exponential mapping of SU(3) in great detail. The calculations depend in some points on the fact that we consider a special group, i.e. the sum over the eigenvalues of the generator vanishes. There is no conceptional problem to extend the method to U(3). As in the other cases (cf [1, 2]), a typical element $U \in \mathrm{SU}(3)$ can be written as an exponential of the generator

$$U = \mathrm{e}^H = \sum_{n=0}^{\infty} \frac{1}{n!} H^n \qquad \text{with } H \in \mathfrak{su}(3). \tag{1}$$

The Cayley–Hamilton theorem and the iterated form in this case read

$$H^3 = b_0 H + c_0 \qquad \text{and} \qquad H^{3+i} = a_i H^2 + b_i H + c_i$$

where the coefficients b_0 and c_0 are functions of the eigenvalues of the eigenvalues x, y, z of H. They satisfy the recurrence relations

$$a_{i+1} = b_i \qquad b_{i+1} = a_i b_0 + c_i \qquad c_{i+1} = a_i c_0. \tag{2}$$

Hence the coefficients a_i satisfy

$$a_{i+1} = a_{i-1} b_0 + a_{i-2} c_0 \tag{3}$$

with the first few values

$$a_0 = 0 \qquad a_1 = b_0 \qquad a_2 = c_0 \qquad a_3 = b_0^2 \qquad a_4 = b_0 c_0.$$

The explicit form of b_0 and c_0 can easily be derived from the secular equation

$$\begin{aligned} 0 &= (\lambda - x)(\lambda - y)(\lambda - z) \\ &= \lambda^3 - (\underbrace{x + y + z}_{a_0 = 0})\lambda^2 + (\underbrace{xy + xz + yz}_{-b_0})\lambda - \underbrace{xyz}_{c_0}. \end{aligned}$$

The leading coefficient a_0 vanishes since the generator H is traceless, i.e. $x + y + z = 0$. The second coefficient can also be written as $b_0 = \frac{1}{2}(x^2 + y^2 + z^2)$. There are also some nice relations

$$b_0 x + c_0 = x^3 \qquad b_0 y + c_0 = y^3 \qquad b_0 z + c_0 = z^3.$$

The idea now is to use the Cayley–Hamilton theorem for writing the sum (1) as

$$\begin{aligned} U &= I_3 + H + \frac{1}{2} H^2 + \sum_{n=0}^{\infty} \frac{1}{(n+3)!} H^{n+3} \\ &= I_3 + H + \frac{1}{2} H^2 + \sum_{n=0}^{\infty} \frac{1}{(n+3)!} (a_n H^2 + b_n H + c_n). \end{aligned}$$

This form contains only the sum over rational functions, there are no longer higher powers of the generator present. The next step is now to find an analytic expression for the sums over the coefficients.

A convenient form for the functions a_n, b_n, and c_n can be obtained if we introduce the multiplier

$$m = (x - y)(x - z)(y - z) = (x^2(y - z) + y^2(z - x) + z^2(x - y)).$$

Then we obtain for the group element

$$mU = m\left(I_3 + H + \frac{1}{2}H^2\right) + \left[\sum_{n=0}^{\infty}\frac{ma_n}{(n+3)!}\right]H^2 + \left[\sum_{n=0}^{\infty}\frac{mb_n}{(n+3)!}\right]H + \left[\sum_{n=0}^{\infty}\frac{mc_n}{(n+3)!}\right].$$

It can easily be shown that the following form for the coefficients satisfy the recurrence relations (2) and (3)

$$\begin{aligned} ma_n &= (y-z)x^{n+3} + (z-x)y^{n+3} + (x-y)z^{n+3} \\ mb_n &= (y-z)x^{n+4} + (z-x)y^{n+4} + (x-y)z^{n+4} \\ mc_n &= yz(y-z)x^{n+3} + xz(z-x)y^{n+3} + xy(x-y)z^{n+3}. \end{aligned} \tag{4}$$

The three sums are now

$$\begin{aligned} \left[\sum_{n=0}^{\infty}\frac{ma_n}{(n+3)!}\right] &= (y-z)\,\mathrm{e}^x + (z-x)\,\mathrm{e}^y + (x-y)\,\mathrm{e}^z - \frac{1}{2}m \\ \left[\sum_{n=0}^{\infty}\frac{mb_n}{(n+3)!}\right] &= x(y-z)\,\mathrm{e}^x + y(z-x)\,\mathrm{e}^y + z(x-y)\,\mathrm{e}^z - m \\ \left[\sum_{n=0}^{\infty}\frac{mc_n}{(n+3)!}\right] &= yz(y-z)\,\mathrm{e}^x + xz(z-x)\,\mathrm{e}^y + xy(x-y)\,\mathrm{e}^z. \end{aligned}$$

Finally, we get the expansion of a SU(3) group element:

$$\begin{aligned} mU = {} & [yz(y-z)\,\mathrm{e}^x + xz(z-x)\,\mathrm{e}^y + xy(x-y)\,\mathrm{e}^z]I_3 \\ & + [x(y-z)\,\mathrm{e}^x + y(z-x)\,\mathrm{e}^y + z(x-y)\,\mathrm{e}^z]H \\ & + [(y-z)\,\mathrm{e}^x + (z-x)\,\mathrm{e}^y + (x-y)\,\mathrm{e}^z]H^2. \end{aligned} \tag{5}$$

2.1. Equal eigenvalues

In the case when some eigenvalues of H are equal, the multiplier m vanishes. To resolve this problem we need to find expressions for the solutions of the recurrence relations (2) which do not involve this multiplier in the original form. What we can do is to rewrite the solutions (4) with a multiplier $\tilde{m}$ which does not vanish if two eigenvalues are equal.

Without loss of generality, let us consider the case where $x = y$. This means that we should eliminate the terms $(x-y)$ from the multiplier m. As an example, let us take the expression for ma_n in equation (4):

$$\begin{aligned} ma_n &= (y-z)x^{n+3} + (z-x)y^{n+3} + (x-y)z^{n+3} \\ &= z(y^{n+3} - x^{n+3}) + xy(x^{n+2} - y^{n+2}) + (z-y)z^{n+3} \\ &= (x-y)(-z(x^{n+2} + x^{n+1}y + \cdots + y^{n+2}) \\ &\quad + xy(x^{n+1} + x^n y + \cdots + y^n) + z^{n+3}). \end{aligned} \tag{6}$$

With the definition

$$m = (x-y)\tilde{m} \qquad \tilde{m} = (x-z)(y-z)$$

we get a solution of the recurrence relations (4):

$$\tilde{m}a_n = -z(x^{n+2} + x^{n+1}y + \cdots + y^{n+2}) + xy(x^{n+1} + x^n y + \cdots + y^n) + z^{n+3}. \tag{7}$$

If we now look at the case $x = y$ we obtain

$$\tilde{m}a_n = -(n+3)zx^{n+2} + (n+2)x^{n+3} + z^{n+3}.$$

The sum over n of these expressions is easily performed:

$$\sum_{n=0}^{\infty} \frac{\tilde{m} a_n}{(n+3)!} = (x-z) \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} - \sum_{n=0}^{\infty} \frac{x^{n+3}}{(n+3)!} + \sum_{n=0}^{\infty} \frac{z^{n+3}}{(n+3)!}$$
$$= (x-z-1)\,\mathrm{e}^x + \mathrm{e}^z - \frac{1}{2} \underbrace{(xz)^2}_{\tilde{m}}. \tag{8}$$

For the other two sums the same method works and we find as a result

$$\tilde{m} b_n = (n+3)(x-z)x^{n+3} - z x^{n+3} + z^{n+4} \tag{9}$$
$$\tilde{m} c_n = (n+3)(xz - z^2)x^{n+3} + (z^2 - 2xz)x^{n+3} + x^2 z^{n+3}. \tag{10}$$

The sums are then

$$\sum_{n=0}^{\infty} \frac{\tilde{m} b_n}{(n+3)!} = (z^2 - x^2 - z)\,\mathrm{e}^x + z\,\mathrm{e}^z - \tilde{m} \tag{11}$$
$$\sum_{n=0}^{\infty} \frac{\tilde{m} c_n}{(n+3)!} = z(x^2 - xz - 2x + z)\,\mathrm{e}^x + x^2\,\mathrm{e}^z - \tilde{m}. \tag{12}$$

If we take into account that we are dealing with a special group we also obtain the relation $z = -2x$. Then the final result is

$$\tilde{m}\,\mathrm{e}^H = ((3x-1)\,\mathrm{e}^x + \mathrm{e}^{-2x})H^2 + x((3x+2)\,\mathrm{e}^x - 2\,\mathrm{e}^{-2x})H + x^2(4(2-3x)\,\mathrm{e}^x + \mathrm{e}^{-2x})I_3. \tag{13}$$

3. The exponential map of GL(N)

As we have seen in the cases of the groups SU(3) and SU(2, 2) [2] the exponential map can be written as a sum over the first $(N-1)$ powers of the generator $H \in \mathfrak{su}(N)$, where the coefficients are functions of the eigenvalues of H. In this section we generalize the results we have found for the low-dimensional examples. It seems that there is a relatively easy concept of generalization.

The desired result is an expansion of the exponential map of the form

$$g = \mathrm{e}^H = \sum_{n=0}^{\infty} \frac{H^n}{n!} = \sum_{k=0}^{N-1} A_k H^k \tag{14}$$

where the coefficients A_k are rational functions of the eigenvalues $\{\lambda_i;\, i = 1, 2, \ldots, N\}$ of the generator H.

3.1. The secular equation

The first step will be to take a look at the eigenvalues, some auxiliary functions, and their inter-relations.

Let us consider the secular equation of the matrix H

$$0 = \prod_{i=1}^{N} (\lambda - \lambda_i) = -\left(\sum_{k=0}^{N} C_k \lambda^{N-k} \right) \tag{15}$$

where the coefficients C_k are functions of the eigenvalues of H. In section 3.5 some coefficients are listed in their explicit form. For later convenience we also introduce the 'truncated' version $C_{(i)k}$ of the coefficients C_k, defined by

$$\prod_{j\neq i}(\lambda-\lambda_j) =: -\sum_{k=0}^{N-1} C_{(i)k}\lambda^{N-1-k}. \tag{16}$$

Essentially, $C_{(i)k}$ contain all terms of C_k without λ_i. The connection between these coefficients can be seen easily via

$$\begin{aligned}\prod_{i=1}^{N}(\lambda-\lambda_i) = (\lambda-\lambda_1)\prod_{i=2}^{N}(\lambda-\lambda_i) &= (\lambda-\lambda_1)\left(\lambda^{N-1}-\sum_{k=1}^{N-1} C_{(1)k}\lambda^{N-1-k}\right)\\ &= \lambda^N+\lambda_1 C_{(1)N-1}-\sum_{k=0}^{N-2}[C_{(1)k+1}-\lambda_1 C_{(1)k}]\lambda^{N-1-k}\\ &\overset{!}{=} \lambda^N-\sum_{k=0}^{N-1} C_{k+1}\lambda^{N-1-k}.\end{aligned}$$

Since the calculations above can be generalized to all eigenvalues we have the relations

$$\begin{aligned}&C_k = C_{(i)k}-\lambda_i C_{(i)k-1} \qquad \text{for } k=1,2,\ldots,N-1\\ &C_N = -\lambda_i C_{(i)N-1}.\end{aligned} \tag{17}$$

Let us define the multiplier m; i.e. the discriminant of the secular equation

$$m := \prod_{i<j}(\lambda_i-\lambda_j) \tag{18}$$

and the functions (see also (39))

$$m_i := m(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_N) = \prod_{\substack{j<k\\ j,k\neq i}}(\lambda_j-\lambda_k). \tag{19}$$

In what follows we mainly use the following form of m which can be obtained by expanding the Slater determinant (see (35) and (38)):

$$m = \sum_{i=1}^{N}(-1)^{i+1} m_i {\lambda_i}^{N-1}. \tag{20}$$

This formula can be generalized to

$$m\delta_{kl} = \sum_{i=1}^{N}(-1)^{i} m_i C_{(i)k-1}\lambda_i^{N-l} \qquad \text{for } k,l=1,2,\ldots,N. \tag{21}$$

For the proof see section 3.4.

3.2. *Recurrence relations*

The described method relies on the Cayley–Hamilton theorem which gives us the ability to write all powers H^{N+n} for $n\in\mathbb{N}$ in terms of the first $N-1$ powers of H. The Cayley–Hamilton theorem for $H\in\mathfrak{gl}(N)$ reads

$$H^N = \sum_{k=1}^{N} C_k H^{N-k}. \tag{22}$$

The coefficients C_k are the same as those in the secular equation (15) and satisfy the recurrence relations (24) derived below. For the special groups, i.e. $\det \boldsymbol{g} = 1$ for $g \in \mathrm{SL}(N)$ the first coefficient vanishes since the sum over the eigenvalues is zero.

Multiplication of (22) with H^n and using (22) again gives the iterated form

$$H^{N+n} = \sum_{k=1}^{N} C_k^n H^{N-k}. \tag{23}$$

Multiplying once more with H gives

$$\begin{aligned} H^{N+n+1} &= (C_2^n + C_1^n C_1) H^{N-1} + (C_3^n + C_1^n C_2) H^{N-2} + \cdots + (C_{n+1}^n + C_1^n C_n) H^{N-n} + \cdots \\ &\quad \cdots + (C_N^n + C_1^n C_{N-1}) H + C_1^n C_N \\ &\stackrel{!}{=} \sum_{k=1}^{N} C_k^{n+1} H^{N-k} \end{aligned}$$

and hence we obtain the recurrence relations

$$\begin{aligned} &C_1^{n+1} = C_2^n + C_1^n C_1 \qquad C_2^{n+1} = C_3^n + C_1^n C_2 \\ &\ldots C_k^{n+1} = C_{k+1}^n + C_1^n C_k \ldots \\ &C_{N-1}^{n+1} = C_N^n + C_1^n C_{N-1} \qquad C_N^{n+1} = C_1^n C_N. \end{aligned} \tag{24}$$

If we successively plug in C_k^j in the recurrence relation of C_1^n we find a formula which contains only terms with C_1^n and the coefficients of the original Cayley–Hamilton equation (22)

$$C_1^{n+1} = \begin{cases} \displaystyle\sum_{j=1}^{N} C_1^{n+1-j} C_j & \text{for } n \geqslant N-1 \\ \displaystyle\sum_{j=0}^{n} C_1^{n-j} C_{1+j} + mC_{n+2} & \text{for } n < N-1. \end{cases} \tag{25}$$

For the other coefficients $C_k^{n+1} (k = 1, 2, \ldots, N)$ we have analogous formulae

$$C_k^{n+1} = \begin{cases} \displaystyle\sum_{j=0}^{N-k} C_1^{n-j} C_{k+j} & \text{for } n \geqslant N-k \\ \displaystyle\sum_{j=0}^{n} C_1^{n-j} C_{k+j} + mC_{k+n+1} & \text{for } n < N-k. \end{cases} \tag{26}$$

The coefficients of the secular equation have the explicit form

$$\begin{aligned} mC_k &\stackrel{(20)}{=} \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} C_k \\ &\stackrel{(17)}{=} \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} (C_{(i)k} - \lambda_i C_{(i)k-1}) \\ &= \; -\sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N} C_{(i)k-1} + \underbrace{\sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} C_{(i)k}}_{=0 \text{ for } k \neq 0} \end{aligned}$$

where we have applied equation (43) to the second term in the last equation. We will need this form as first values in the proof of equation (30):

$$mC_k = \sum_{i=1}^{N}(-1)^i m_i \lambda_i^N C_{(i)k-1} \qquad \text{for } k = 1, 2, \ldots, N. \tag{27}$$

From the SU(3) and SU(2, 2) cases one may assume that the recurrence relation (25) has the solution

$$mC_1^n = \sum_{i=1}^{N}(-1)^{i+1} m_i \lambda_i^{N+n}. \tag{28}$$

Proof. The proof of equation (28) is shown by induction over n.

The first coefficient ($n = 0$) is given by

$$mC_1 = \sum_{i=1}^{N}(-1)^{i+1} m_i \lambda_i^N \tag{29}$$

which is easy to prove if one writes C_1 in the form

$$C_1 = \sum_{j=1}^{N}\lambda_j = \lambda_i + \sum_{j\neq i}\lambda_j = \lambda_i + C_{(i)1}.$$

For the product mC_1 we take m in the form of equation (38),

$$\begin{aligned} mC_1 &= \sum_{i=1}^{N}(-1)^{i+1} m_i \lambda_i^{N-1}(\lambda_i + C_{(i)1}) \\ &= \sum_{i=1}^{N}(-1)^{i+1} m_i \lambda_i^{N} + \underbrace{\sum_{i=1}^{N}(-1)^{i+1} m_i C_{(i)1}\lambda_i^{N-1}}_{=0}. \end{aligned}$$

The last equation holds since the exponent of λ_i should be $N - 2$ in order to yield a non-vanishing sum (see equation (43)).

First we treat the case of $n \geqslant N$. The next step is to assume the validity of (28) for n and to show that then it follows also for $n + 1$:

$$\begin{aligned} mC_1^{n+1} &\overset{(25)}{=} \sum_{j=1}^{N} mC_1^{n+1-j} C_j \overset{(28)}{=} \sum_{j=1}^{N}\sum_{i=1}^{N}(-1)^{i+1} m_i \lambda_i^{N+n+1-j} C_j \\ &\overset{(17)}{=} \sum_{j=1}^{N}\sum_{i=1}^{N}(-1)^{i+1} m_i \lambda_i^{N+n+1-j}(C_{(i)j} - \lambda_i C_{(i)j-1}) \\ &= \sum_{i=1}^{N}(-1)^{i+1} m_i \left(\sum_{j=1}^{N} C_{(i)j}\lambda_i^{N+n+1-j} - \sum_{j=1}^{N} C_{(i)j-1}\lambda_i^{N+n+2-j}\right) \\ &= \sum_{i=1}^{N}(-1)^{i+1} m_i \left(\sum_{j=2}^{N+1} C_{(i)j-1}\lambda_i^{N+n+2-j} - \sum_{j=1}^{N} C_{(i)j-1}\lambda_i^{N+n+2-j}\right) \\ &= \sum_{i=1}^{N}(-1)^{i+1} m_i \left(\underbrace{C_{(i)N}}_{=0}\lambda_i^{n+1} - \underbrace{C_{(i)0}}_{=-1}\lambda_i^{N+n+1}\right) \\ &= \sum_{i=1}^{N}(-1)^{i+1} m_i \lambda_i^{N+n+1} = mC_1^{n+1}. \end{aligned}$$

In the case of $n < N-1$ there is an additional term

$$\begin{aligned}
mC_1^{n+1} &= \sum_{j=0}^{n} mC_1^{n-j} C_{1+j} + mC_{n+2} = \cdots \\
&= \sum_{i=1}^{N} (-1)^{i+1} m_i (C_{(i)n+1}\lambda_i^N - C_{(i)0}\lambda_i^{N+n+1}) + mC_{n+2} \\
&= \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n+1} + \underbrace{\sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)n+1}\lambda_i^N}_{=-mC_{n+2}} + mC_{n+2} \\
&\overset{(27)}{=} \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n+1}.
\end{aligned}$$

□

The coefficients C_k^n can be written in the form

$$mC_k^n = \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1}\lambda_i^{N+n} \qquad \text{for } k = 1, \ldots, N. \tag{30}$$

Proof. The proof is analogous to the one for mC_1^n but uses the explicit form (28) of these coefficients:

$$\begin{aligned}
mC_k^{n+1} &= \sum_{j=0}^{N-k} mC_1^{n-j} C_{k+j} \overset{(28)}{=} \sum_{j=0}^{N-k} \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n-j} C_{k+j} \\
&\overset{(17)}{=} \sum_{j=0}^{N-k} \sum_{i=1}^{N} (-1)^{i+1} m_i (C_{(i)k+j} - \lambda_i C_{(i)k+j-1})\lambda_i^{N+n-j} \\
&= \sum_{j=0}^{N-k} \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)k+j} \lambda_i^{N+n-j} \\
&\quad - \sum_{j=1}^{N-k} \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)} k + j - 1 \lambda_i^{N+1+n-j} - \sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)k-1}\lambda_i^{N+n+1}.
\end{aligned}$$

If we now shift the summation index j in the second sum most of the terms cancel with those of the first sum. The remaining term $j = N-k$ in the first sum contains $C_{(i)N} = 0$ and, hence, vanishes also. Therefore, only the third sum remains to proof the assumed form (30) of C_k^n.

Again there are the cases $n < N-k$ which need to be treated separately:

$$\begin{aligned}
mC_k^{n+1} &= \sum_{j=0}^{N-k} mC_1^{n-j} C_{k+j} + mC_{k+n+1} = \cdots \\
&= \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1}\lambda_i^{N+n+1} + \underbrace{\sum_{i=1}^{N} (-1)^{i+1} m_i C_{(i)k+n}\lambda_i^N}_{=-mC_{k+n+1} (27)} + mC_{k+n+1} \\
&= \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1}\lambda_i^{N+n+1}.
\end{aligned}$$

□

3.3. *The exponential map*

The expansion of a group element $\boldsymbol{g} \in \boldsymbol{G}$ with generator $H \in \mathfrak{gl}(N)$ can now be written as

$$\begin{aligned}\boldsymbol{g} &= \mathrm{e}^{H} \\ &= \sum_{n=0}^{\infty} \frac{H^n}{n!} \\ &= \sum_{n=0}^{N-1} \frac{H^n}{n!} + \sum_{n=0}^{\infty} \frac{H^{N+n}}{(N+n)!} \\ &= \sum_{n=0}^{N-1} \frac{H^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{(N+n)!} \left(\sum_{k=1}^{N} C_k^n H^{N-k} \right). \end{aligned} \tag{31}$$

Using the multiplier m we get

$$m\boldsymbol{g} = m \sum_{n=0}^{N-1} \frac{H^n}{n!} + \sum_{k=1}^{N} \left[\sum_{n=0}^{\infty} \frac{1}{(N+n)!} m C_k^n \right] H^{N-k}. \tag{32}$$

We can now treat the sums for different k separately:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(N+n)!} m C_k^n &= \sum_{n=0}^{\infty} \frac{1}{(N+n)!} \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1} \lambda_i^{N+n} \\ &= \sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i \sum_{n=0}^{\infty} \frac{1}{(N+n)!} \lambda_i^{N+n} \\ &= \sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i \left(\mathrm{e}^{\lambda_i} - \sum_{n=0}^{N-1} \frac{\lambda_i^n}{n!} \right) \\ &= \sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i \, \mathrm{e}^{\lambda_i} + \sum_{n=0}^{N-1} \frac{1}{n!} \sum_{i=1}^{N} (-1)^{i+1} C_{(i)k-1} m_i \lambda_i^n \\ &= \sum_{i=1}^{N} (-1)^i C_{(i)k-1} m_i \, \mathrm{e}^{\lambda_i} - \frac{m}{(N-k)!}. \end{aligned}$$

The last equation relies on equations (42) and (43). The terms $-m/(N-k)!$ cancel the first sum in equation (32).

The final result turns out to be

$$\begin{aligned} m\,\mathrm{e}^{H} = (-1)^N \det H & \left(\sum_{i=1}^{N} (-1)^i m_i \frac{\mathrm{e}^{\lambda_i}}{\lambda_i} \right) I_N + \left(\sum_{i=1}^{N} (-1)^i C_{(i)N-2} m_i \, \mathrm{e}^{\lambda_i} \right) H + \cdots \\ & \cdots + \left(\sum_{i=1}^{N} (-1)^i C_{(i)k} m_i \, \mathrm{e}^{\lambda_i} \right) H^{N-1-k} + \cdots \\ & \cdots + \left(\sum_{i=1}^{N} (-1)^i m_i \lambda_i \, \mathrm{e}^{\lambda_i} \right) H^{N-2} + \left(\sum_{i=1}^{N} (-1)^{i+1} m_i \mathrm{e}^{\lambda_i} \right) H^{N-1} \end{aligned}$$

or in closed form

$$m\,\mathrm{e}^{H} = \sum_{n=1}^{N} \left(\sum_{i=1}^{N} (-1)^i C_{(i)n-1} m_i \, \mathrm{e}^{\lambda_i} \right) H^{N-n} \tag{33}$$

which reads in terms of the adjoints of the Slater determinant

$$m\,\mathrm{e}^{H} = \sum_{n=0}^{N-1}\left(\sum_{i=1}^{N} A_{(i)n}\,\mathrm{e}^{\lambda_i}\right)H^{n}. \tag{34}$$

3.4. The Slater determinant

One crucial ingredient of the method is the usage of a multiplier m, defined in equation (18). From low-dimensional examples one may assume the forms (20) and (21) of m. The general proofs can be performed by writing m as a Slater determinant. The Slater determinant is defined as (cf [6])

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_N & \lambda_{N-1} & \dots & \lambda_1 \\ \lambda_N^2 & \lambda_{N-1}^2 & \dots & \lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N^{N-1} & \lambda_{N-1}^{N-1} & \dots & \lambda_1^{N-1} \end{vmatrix} = \prod_{i<j}(\lambda_i - \lambda_j) = m. \tag{35}$$

We can now use the Laplacian method of expanding the Slater determinant

$$\det A = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n} a_{ij}A_{ij} \tag{36}$$

where the so-called adjoints A_{ij} are the subdeterminants of a_{ij} multiplied by the sign factor $(-1)^{i+j}$.

It is also well known that the Laplace expansion with 'wrong' adjoints gives zero

$$0 = \sum_{j=1}^{n} a_{ij}A_{lj} \qquad \text{for } l \neq i. \tag{37}$$

The Laplacian method applied with respect to the last row then gives the expansion (20),

$$m = \sum_{i=1}^{N} \lambda_i^{N-1} A_{(N+1-i)N} = \sum_{i=1}^{N} (-1)^{N+(N+1-i)} m_i \lambda_i^{N-1} = \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N-1} \tag{38}$$

where m_i are the subdeterminants of m

$$m_i = \prod_{\substack{k<j \\ k,j\neq i}} (\lambda_k - \lambda_j). \tag{39}$$

We have

$$\begin{aligned} m &= m_i \prod_{k<i}(\lambda_k - \lambda_i)\prod_{i<j}(\lambda_i - \lambda_j) \qquad \text{for } i = 1, 2, \dots, N \\ &= (-1)^{i-1} m_i \prod_{j\neq i}(\lambda_i - \lambda_j) \\ &\overset{(16)}{=} (-1)^{i-1} m_i \sum_{n=0}^{N-1}\left(-C_{(i)n}\lambda_i^{N-1-n}\right) \\ &= \sum_{n=0}^{N-1} (-1)^{i} m_i C_{(i)N-1-n}\lambda_i^{n} \overset{(36)}{=} \sum_{n=0}^{N-1} A_{(i)n}\lambda_i^{n} \end{aligned} \tag{40}$$

which proves equation (42). Also equation (43) is proven since if the exponent of λ_i is not n the sum vanishes because it is an expansion with the 'wrong' adjoints $A_{(i)n}$. Hence, we get an explicit expression for the adjoints

$$A_{(i)n} = (-1)^i m_i C_{(i)N-n-1}. \tag{41}$$

We can also expand m with respect to the $(n+1)$th line and then use equation (41):

$$\begin{aligned} m &= \sum_{i=1}^{N} A_{(i)n}\lambda_i^n \qquad \text{for } n = 0, 1, \ldots, N-1 \\ &= \sum_{i=1}^{N} (-1)^i m_i C_{(i)N-1-n}\lambda_i^n. \end{aligned} \tag{42}$$

Writing the Laplacian expansion with 'wrong' adjoints leads to

$$\begin{aligned} 0 &= \sum_{i=1}^{N} A_{(i)k}\lambda_i^n \qquad \text{for } k, n = 0, 1, \ldots, N-1, k \neq n \\ &= \sum_{i=1}^{N} (-1)^i m_i C_{(i)N-1-k}\lambda_i^n. \end{aligned} \tag{43}$$

Ergo (cf equation (21))

$$m\delta_{kl} = \sum_{i=1}^{N} (-1)^i m_i C_{(i)k-1}\lambda_i^{N-l} \qquad \text{for } k, l = 1, 2, \ldots, N. \tag{44}$$

3.5. Case of two equal eigenvalues

In the cases where some of the eigenvalues coincide, the multiplier m will be zero. However, in these cases m can be chosen in a simpler non-vanishing form.

Here we show how the recurrence relations (26) can solved if two eigenvalues of the general $\mathfrak{gl}(N)$ matrix H are equal. It is a straightforward generalization of the calculations of the case for SU(3).

Let us assume that the two eigenvalues λ_1 and λ_2 will become equal. As in the previous $(N = 3)$ case we shall try to extract the factor $(\lambda_1 - \lambda_2)$ from the solutions (30) of the recurrence relations (26). Then we can use a multiplier $\tilde{m}$ which does not contain the later vanishing factor $(\lambda_1 - \lambda_2)$.

Let us first consider the coefficients C_1^n. The new multipliers $\tilde{m}$ and $\tilde{m}_i$ are defined to be

$$m = (\lambda_1 - \lambda_2)\tilde{m} \qquad \text{and} \qquad m_i = (\lambda_1 - \lambda_2)\tilde{m}_i.$$

Starting from equation (28) we extract $(\lambda_1 - \lambda_2)$ from C_1^n:

$$\begin{aligned} mC_1^n &= \sum_{i=1}^{N} (-1)^{i+1} m_i \lambda_i^{N+n} \\ &= \left(\lambda_1^{N+n} \prod_{k=3}^{N} (\lambda_2 - \lambda_k) - \lambda_2^{N+n} \prod_{k=3}^{N} (\lambda_1 - \lambda_k)\right) \prod_{3 \leqslant k < l}^{N} (\lambda_k - \lambda_l) \\ &\quad + (\lambda_1 + \lambda_2) \sum_{i=3}^{N} (-1)^{i+1} \tilde{m}_i \lambda_i^{N+n} \end{aligned}$$

$$= \left[\sum_{l=0}^{N-2} a_l (\lambda_1^{N+n}\lambda_2^l - \lambda_1^l \lambda_2^{N+n}) \right] \prod_{3\leqslant k<l}^{N} (\lambda_k - \lambda_l)$$

$$+(\lambda_1 + \lambda_2) \sum_{i=3}^{N} (-1)^{i+1} \tilde{m}_i \lambda_i^{N+n} = (\lambda_1 - \lambda_2) \left[\sum_{i=3}^{N} (-1)^{i+1} \tilde{m}_i \lambda_i^{N+n} \right.$$

$$\left. + \left(\sum_{l=0}^{N-2} a_l \lambda_1^l \lambda_2^l \sum_{k=0}^{N+n-1-l} \lambda_1^{N+n-1-l-k} \lambda_2^k \right) \prod_{3\leqslant l<m}^{N} (\lambda_l - \lambda_m) \right] \tag{45}$$

where $\prod_{k=3}^{N}(\lambda - \lambda_k) := \sum_{l=0}^{N-2} a_l \lambda^l$.

In the equation above we can cancel the factor $(\lambda_1 - \lambda_2)$. Hence we can set $\lambda_1 = \lambda_2$ without getting a vanishing multiplier $\tilde{m}$. Now we obtain

$$\tilde{m} C_1^n = \sum_{i=3}^{N} (-1)^{i+1} \tilde{m}_i \lambda_1^{N+n} + \left(\sum_{l=0}^{N-2} a_l \lambda_1^{2l} (N + n - l) \lambda_1^{N+n-1-l} \right) \prod_{3\leqslant l<m}^{N} (\lambda_l - \lambda_m)$$

$$= \sum_{i=3}^{N} (-1)^{i+1} \tilde{m}_i \lambda_i^{N+n} + \left(\frac{\partial}{\partial \lambda_2} \lambda_2^{N+n} \right) m_1 - \lambda_2^{N+n} \frac{\partial}{\partial \lambda_2} m_1. \tag{46}$$

For the other coefficients C_K^n the corresponding calculations are more involved. Again we start from the solution (30) of the recurrence relations

$$m C_k^n = -m_1 C_{(1)k-1} \lambda_1^{N+n} + m_2 C_{(2)k-1} \lambda_2^{N+n} + \sum_{i=3}^{N} (-1)^1 m_i C_{(i)k-1} \lambda_i^{N+n}.$$

Now we need to find a relation between the two series of factors $C_{(1)k}$ and $C_{(2)k}$. Useful relations can be derived by looking at

$$\prod_{j\neq i} (\lambda_i - \lambda_j) = -\sum_{k=0}^{N-1} C_{(i)k} \lambda_i^{N-1-k} \qquad \text{for } i = 1, 2$$

$$= (-1)^{i+1} (\lambda_1 - \lambda_2) \underbrace{\prod_{l=3}^{N} (\lambda_i - \lambda_j)}_{=:\sum_{l=0}^{N-2} B_l \lambda_i^{N-2-l}} . \tag{47}$$

With some arrangements of the sums we obtain the desired relations

$$C_{(2)k} = (\lambda_1 - \lambda_2) B_{k-1} + C_{(1)k} \qquad \text{for } k = 1, \ldots, N-1$$
$$C_{(1)0} = C_{(2)0} = B_0.$$

Using these relations and cancelling the factor $(\lambda_1 - \lambda_2)$ we get

$$\tilde{m} C_k^n = \sum_{i=3}^{N} (-1)^i \tilde{m}_i C_{(i)k-1} \lambda_i^{N+n} - C_{(1)k-1} \left[\sum_{l=0}^{N-2} a_l \lambda_1^l \lambda_2^l \sum_{k=0}^{N+n-1-l} \lambda_1^{N+n-1-l-k} \lambda_2^k \right] \prod_{3\leqslant k<l}^{N} (\lambda_k - \lambda_l)$$

$$+ B_{k-2} \lambda_2^{N+n} \prod_{l=3}^{N} (\lambda_1 - \lambda_l) \prod_{3\leqslant k<l}^{N} (\lambda_k - \lambda_l).$$

Now we set $\lambda_1 = \lambda_2$ and eliminate all occurrences of λ_1. This results in

$$\begin{aligned}\tilde{m}C_k^n = \sum_{i=3}^{N}(-1)^i \tilde{m}_i C_{(i)k-1}\lambda_i^{N+n} - C_{(1)k-1}\left[\sum_{l=0}^{N-2} a_l \lambda_2^{2l}(N+n-l)\lambda_2^{N+n-1-l}\right]& \\ \times \prod_{3\leqslant k<l}^{N}(\lambda_k - \lambda_l) + B_{k-2}\lambda_2^{N+n}\prod_{2\leqslant k<l}^{N}(\lambda_k-\lambda_l)& \\ = \sum_{i=3}^{N}(-1)^i \tilde{m}_i C_{(i)k-1}\lambda_i^{N+n} + B_{k-2}m_1\lambda_2^{N+n}& \\ -C_{(1)k-1}m_1\frac{\partial}{\partial\lambda_2}\lambda_2^{N+n} + C_{(1)k-1}\lambda_2^{N+n}\frac{\partial}{\partial\lambda_2}m_1.& \qquad (48)\end{aligned}$$

This equations holds of course for an arbitrary pair of eigenvalues. If there are more equal eigenvalues of H an analogous method can be used to obtain solutions of the recurrence relations, but the calculations and formulae become rather lengthy. If there are more pairs of equal eigenvalues we immediately have the results, but if more than two eigenvalues are equal the calculations becomes nasty. Some of the terms used above have the explicit form

$$\tilde{m} = \prod_{k=3}^{N}(\lambda_1-\lambda_k)\prod_{2\leqslant k<l}^{N}\prod_{2\leqslant k<l}^{N}(\lambda_k-\lambda_l)$$

$$\tilde{m}_i = \prod_{\substack{k=3\\k\neq i}}^{N}(\lambda_1-\lambda_k)\prod_{\substack{2\leqslant k<l\\k,l\neq i}}^{N}(\lambda_k-\lambda_l) \qquad i=3,\ldots,N.$$

4. Conclusion

The method developed here to exponentiate a matrix can also be applied to other convergent power-series expansions of matrix functions $f(H)$. Obviously, the final result can be obtained by replacing e^{λ_i} by $f(\lambda_i)$ in equation (33). These results are similar to those obtained by the Lagrange–Sylvester interpolation (cf [4]).

The considered problem of exponentiating a matrix A is not only interesting by itself but has a wide range of applications. The most prominent one is the fact that the exponential $\exp tA$ is a solution of the first-order differential equation $\dot{x}(t) = Ax(t)$. The possibility to describe finite transformations and to expand explicitly the Hausdorff formula is also important.

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Appendix A. Some details

This section contains explicit forms of some coefficients and some proofs. Almost all of the equations hold in the general case, but those which hold only in the case of the special groups, i.e. vanishing sum of eigenvalues, are denoted by the sign $\stackrel{s}{=}$.

For the coefficients of the secular equation we obtain, for example,

$$C_N = (-1)^{N+1}\prod_{i=1}^{N}\lambda_i = (-1)^{N+1}\det H$$
$$C_{N-1} = (-1)^N\sum_{i=1}^{N}\prod_{j\neq i}\lambda_j = (-1)^N\sum_{i=1}^{N}\frac{\det H}{\lambda_i}$$
$$C_2 = (-1)\sum_{i<j}\lambda_i\lambda_j \stackrel{\mathrm{s}}{=} \frac{1}{2}\sum_{k=1}^{N}{\lambda_k}^2$$
$$C_1 = \sum_{i=1}^{N}\lambda_i \stackrel{\mathrm{s}}{=} 0 \qquad C_0 = -1. \tag{49}$$

Some 'truncated' coefficients are

$$C_{(i)N-1} = (-1)^{N-2}\prod_{j\neq i}\lambda_j = (-1)^N\frac{\det H}{\lambda_i}$$
$$C_{(i)N-2} = (-1)^{N-3}\sum_{k\neq i}\prod_{j\neq k,i}\lambda_j$$
$$C_{(i)2} = (-1)\sum_{\substack{j<k\\ j,k\neq i}}\lambda_j\lambda_k \qquad C_{(i)1} = \sum_{j\neq i}\lambda_j \stackrel{\mathrm{s}}{=} -\lambda_i$$
$$C_{(i)0} = -1 \qquad C_{(i)N} = 0.$$

Appendix B. Additional checks

B.1. One-dimensional subgroups

One-dimensional subgroups (cf [5]) of GL(N) can be generated by

$$\{\mathrm{e}^{tH};\, H \in \mathfrak{gl}(N),\, t \in \mathbb{R}\}. \tag{50}$$

From the known expansion of e^H we can derive the expansion of e^{tH} by multiplying the occurring expressions by an appropriate factor. Obviously, the eigenvalues of the t-dependent generator tH are $t\lambda_i$ if the λ_i are the eigenvalues of H. Therefore, we need to make the replacements

$$\lambda_i \longrightarrow t\lambda_i$$
$$C_k \to t^k C_k \qquad C_{(i)k} \to t^k C_{(i)k}$$
$$m \to t^{N(N-1)/2} m \qquad m_i \to t^{(N-1)(N-2)/2} m_i.$$

The expansion (34) now reads

$$\begin{aligned} t^{N(N-1)/2} m\, \mathrm{e}^{tH} &= \sum_{n=0}^{N-1}\left(\sum_{i=1}^{N}(-1)^i t^{n-1} C_{(i)n-1} t^{(N-1)(N-2)/2} m_i\, \mathrm{e}^{tH}\right)(tH)^{N-n} \\ &= t^{N(N-1)/2}\sum_{n=0}^{N-1}\left(\sum_{i=1}^{N}(-1)^i C_{(i)n-1} m_i\, \mathrm{e}^{tH}\right)H^{N-n} \end{aligned}$$

or

$$m\,\mathrm{e}^{tH} = \sum_{n=0}^{N-1}\left(\sum_{i=1}^{N}(-1)^i C_{(i)n-1} m_i\, \mathrm{e}^{tH}\right)H^{N-n}. \tag{51}$$

Differentiation of the right-hand side of equation (51) and setting $t = 0$ gives the derivation of the unit element

$$\sum_{n=0}^{N-1} \underbrace{\left(\sum_{i=1}^{N} (-1)^i C_{(i)n-1} m_i \lambda_i \right)}_{m\delta_{n,N-1}(\mathrm{cf}(21))} H^{N-n} = mH.$$

Since this result coincides with the one we obtain by differentiating the left-hand side, it is an additional proof of the expansion (33).

B.2. Eigenvalues

It is easy to demonstrate that equation (34) also gives the right connection between the eigenvalues of the generator H and those for the corresponding group element $g = \mathrm{e}^H$. Let x_i be the eigenvectors of H with eigenvalues λ_i

$$Hx_i = \lambda_i x_i \qquad \text{for } i = 1, 2, \ldots, N.$$

For the powers of H we get

$$H^n x_i = \lambda_i^n x_i \qquad \text{for } n \in \mathbb{N}.$$

Plugging this in equation (33) yields

$$\begin{aligned} m\,\mathrm{e}^H x_j &= \sum_{n=0}^{N-1} \left(\sum_{i=1}^{N} A_{(i)n}\, \mathrm{e}^{\lambda_i} \right) H^n x_j = \sum_{n=0}^{N-1} \sum_{i=1}^{N} A_{(i)n}\, \mathrm{e}^{\lambda_i} \lambda_j^n x_j \\ &= \sum_{i=1}^{N} \underbrace{\left(\sum_{n=0}^{N-1} A_{(i)n} \lambda_j^n \right)}_{m\delta_{ij}(\mathrm{cf}(21))} \mathrm{e}^{\lambda_i} x_j = m\mathrm{e}^{\lambda_j} x_j. \end{aligned}$$

Therefore, we obtain the desired result

$$gx_j = \mathrm{e}^{\lambda_j} x_j$$

which again confirms the expansion (33).

Remark. After finishing this work I became aware of a recent paper of Kusnezov [7] on the exact matrix expansion for SU(N). He uses a different method which involves a differential equation and derivatives of the eigenvalues with respect to the group parameters. The present method avoids derivatives and uses only simple calculations. For details see [7].

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